JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 20, No. 1, January–February 1997

Minimum Time Maneuver of Flexible Systems Using Pulse Response Based Control

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The pulse response based control method is applied to minimum time maneuver of flexible structures. When pulse response based control is applied to linear systems, a state-variable representation of the system is not needed, because system behavior is represented in terms of outputs resulting from pulses in control inputs, without modal truncation. However, for maneuver problems, nonlinear behavior associated with rapid large-angle rotation can be significant. A method is presented for taking nonlinear effects into account by explicitly modeling response in only the lowest few flexible modes and using the results to correct the measured pulse response upon which a minimum time control profile is based. A numerical example demonstrates that even for extremely rapid maneuver, nonlinear effects only need to be considered for a very small number of flexible modes, so that requirements for identification and computation are very low.

Introduction

IN the minimum time maneuver of flexible spacecraft, bounded control inputs are used to accomplish a given change in rigid-body modal states as quickly as possible, with minimal vibration at the end of the maneuver. A number of different approaches have been taken to address problems in which nonlinear behavior is significant, and some of these are briefly reviewed here.

Turner and Junkins[1] represent flexible motion in terms of several assumed modes and minimize a quadratic performance index with a single-actuator control, with specified final states and final time. A continuation method is used to obtain a solution of the nonlinear two-point boundary value problem that results when kinematic nonlinearity is present in the formulation. Turner and Chun[2] extend this approach for the case in which a number of actuators are distributed throughout the structure.

Meirovitch and Quinn[3] develop a perturbation approach in which the rigid-body maneuver is taken as a zero-order problem, and small motions, including flexible motions and small rigid-body motions, are treated as a first-order perturbation for nonlinear flexible maneuver problems. The rigid-body maneuver problem is solved using an inverse dynamics approach in which the required zero-order feedforward control torques are obtained from the nonlinear rigid-body equations of motion. The first-order, linear vibration problem with time-varying coefficients is solved independently using feedback control.

Chun et al.[4] obtain a frequency-shaped open-loop control for the rigid-body modes, using a continuation method to handle nonlinearity, and then design a feedback control for the flexible motion by linearizing the flexible response about several points in the open-loop rigid-body trajectory. In later work, they replace the solution of the open-loop problem for the rigid-body modes with a programmed-motion/inverse dynamics approach, where the trajectories of the rigid-body modes are simply specified as smooth functions and the required control torques are obtained from the nonlinear rigid-body equations of motion.[5]

Meirovitch and Sharony[6] use the perturbation approach described earlier and use a minimum time (bang-bang) control to handle the rigid-body problem and a linear quadratic regulator with integral feedback and prescribed convergence rate to drive the flexible motion to zero at the end of the maneuver. Bounds on control inputs are only considered in the design of the rigid-body portion of the control, and so, as in all of the work mentioned thus far, a true minimum time control problem for the flexible maneuver is not solved.

Minimum time control is investigated by Ben-Asher et al.,[7] in which the flexible behavior is modeled in terms of a small number of assumed modes, and switching times for the bang-bang control of this model are found for both linear and nonlinear problems using a parameter optimization approach. More recently, Junkins et al.[8] have pursued an approach in which near minimum time maneuver is achieved by smoothing a bang-bang control profile, and a Lyapunov stable feedback control is designed to suppress the flexible motion that is excited by this profile. This approach has the advantage of requiring very little model data for the system even when nonlinear behavior is present.

There are several factors that contribute to the difficulty of minimum time flexible maneuver problems, and these are all related to modeling the flexible behavior of the spacecraft. First, there is the difficulty of accurate system identification for realistic spacecraft. Second, because the systems to be controlled are distributed parameter systems, they typically require high-order models for representing all significant behavior, and this greatly increases computational requirements associated with control, particularly if a nonlinear two-point boundary value problem must be solved. Third, a finite-order model is guaranteed to have a bang-bang minimum time control, for which only the switching times must be determined. However, the minimum time control for the distributed parameter system being modeled is not ordinarily bang-bang,[9] so that the optimal control profile is qualitatively quite different from the minimum time control of a finite-order model of the system.

Recently, a new approach to minimum time control of flexible spacecraft known as pulse response based control (PRBC) has been developed.[10] For linear problems, a state-variable representation of the system is not required. Pulses in control inputs, of finite amplitude and width, are applied to the system, and sensor measurements of the response to these pulses are recorded. Then sensor measurements of response to a sequence of pulses, i.e., a piecewise constant function of time, in the control input(s) can be obtained by using this response data in a convolution sum. A true minimum time problem is solved by finding a control profile that results in measurements consistent with the desired final state of the system, with a minimal number of time steps. The two-point boundary value problem encountered in a standard optimal control approach is replaced with a numerical optimization problem, for which an efficient algorithm has been developed. This approach has the advantages that pulse response measurements can be obtained much more quickly and easily than the state variable representation required for conventional approaches, using the control hardware that is already required for the mission of the spacecraft. The pulse response measurements contain

Received Oct. 31, 1992; revision received Sept. 3, 1996; accepted for publication Sept. 3, 1996.

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contributions from all of the modes of the system. If the pulses are in the actuator commands, rather than in the actual forces or moments applied to the system, actuator dynamics will be represented in the pulse response as part of the dynamics of the overall system. The precision with which the final state of the linear system can be specified is limited only by the observability of the system with the given outputs. The control history obtained is not bang-bang, and the only known exact solution of a minimum time control problem for a flexible structure, which happens to be piecewise constant in time, is easily obtained using PRBC.[11]

The PRBC method is extended so that nonlinear behavior associated with large-angle rotation in maneuver problems can be handled. Usually nonlinear behavior is significant in only a very small number of flexible modes, and explicit knowledge of these modes is required for the approach presented. However, most of the modes whose response is significant exhibit linear behavior, which is still represented in terms of pulse response so that very few modal data need to be known explicitly. Collocated actuators and sensors are not required in this approach, as they are for methods based on Lyapunov stability theory. The amount of computation required for nonlinear problems is not much greater than the amount required for linear problems.

In the second section, the PRBC method is briefly presented. The extension of the method to nonlinear problems is discussed in the third section. In the fourth section, a numerical example is presented. The final section contains conclusions.

PRBC

Suppose a linear dynamic system is controlled by a single input $u(t)$, with no other excitations, and its response is measured in terms of m outputs that constitute a vector $\boldsymbol{y}(t)$. If the system is initially at rest and a square pulse in $u(t)$ of unit amplitude and duration Δt is applied, and the outputs are sampled every Δt for a total time $n\Delta t$, an $m \times n$ pulse response matrix H is obtained of the form

$$\begin{aligned} H &\equiv \{\boldsymbol{y}(n\Delta t) \quad \boldsymbol{y}[(n-1)\Delta t] \quad \cdots \quad \boldsymbol{y}(\Delta t)\} \\ &= [\boldsymbol{h}_1 \quad \boldsymbol{h}_2 \quad \cdots \quad \boldsymbol{h}_n] \end{aligned} \tag{1}$$

The matrix H is a very simple example of a generalized Hankel matrix. Whereas generalized Hankel matrices are useful in realization methods,[12] recall that the method presented here is intended to minimize the need for system identification.

If the system is initially at rest and a control profile $u(t)$ that is constant over each time step Δt between $t = 0$ and $t = n\Delta t$ is applied to the system, so that the control input is equal to u_i over the ith time step, the output vector at $n\Delta t$ can be found by superposition, in terms of a convolution sum. Each value u_i is multiplied by a vector in the pulse response matrix H just given, so that the output vector is given by

$$\boldsymbol{y}(n\Delta t) = \sum_{i=1}^{n} \boldsymbol{h}_i u_i = H\boldsymbol{u} \tag{2}$$

where $\boldsymbol{u} = [u_1 \ u_2 \ \cdots \ u_n]^T$.

If there are p inputs, the entries u_i in the vector $\boldsymbol{u}$ are replaced by $\boldsymbol{u}_i = [u_{1,i} \ \cdots \ u_{p,i}]^T$ and the vectors $\boldsymbol{h}_i$ must be replaced by $m \times p$ submatrices H_i. The pulse response matrix H must then be filled by applying pulses in the control inputs one at a time and sampling the outputs resulting from each pulse every Δt for $0 < t \leq n\Delta t$. The outputs resulting from a pulse in the first control input will fill the first vector of each submatrix H_i in H, and each of the other $p - 1$ columns in a submatrix H_i will be obtained from response to a pulse in another control input.

If a control task, which consists of driving the system from rest at $t = 0$ to a desired state at a time $t_f = n\Delta t$, is successfully accomplished by a control profile of the type just described, the set of linear equations

$$\boldsymbol{y}(t_f) = H\boldsymbol{u} \tag{3}$$

must be satisfied by the control profile $\boldsymbol{u}$, where the vector $\boldsymbol{y}(t_f)$ contains outputs consistent with the desired final state of the system. Also, if there is no excitation of the system after t_f, the outputs at a time $t_f + j\Delta t$ must be given by the equations

$$\boldsymbol{y}(t_f + j\Delta t) = [H_{1-j} \quad \cdots \quad H_{n-j}]\boldsymbol{u} \tag{4}$$

where the time-shifted submatrix H_{1-j} contains outputs measured $n + j$ time steps after applying test pulses in control inputs, for example, and where the outputs $\boldsymbol{y}(t_f + j\Delta t)$ must be consistent with the desired state of the system at t_f, with free response after t_f.

These observations suggest the following approach to a minimum time control problem when control inputs are bounded. Find the smallest integer n such that there exists a control history vector $\boldsymbol{u} \equiv [\boldsymbol{u}_1^T \ \cdots \ \boldsymbol{u}_n^T]^T$ satisfying the equations

$$\bar{H}\boldsymbol{u} = \bar{\boldsymbol{y}}_f \tag{5}$$

and the input bounds, typically of the form

$$B_j^l \leq u_{j,i} \leq B_j^u, \qquad j = 1, \ldots, p; \qquad i = 1, \ldots, n \tag{6}$$

in which B_j^l and B_j^u are lower and upper bounds for the jth control input. The vector $\bar{\boldsymbol{y}}_f$ contains outputs at t_f and at l time steps afterward:

$$\bar{\boldsymbol{y}}_f \equiv \left\{ \begin{array}{c} \boldsymbol{y}(t_f) \\ \boldsymbol{y}(t_f + \Delta t) \\ \vdots \\ \boldsymbol{y}(t_f + l\Delta t) \end{array} \right\} \tag{7}$$

and the matrix $\bar{H}$ is given by

$$\bar{H} \equiv \begin{bmatrix} H_1 & H_2 & \cdots & H_n \\ H_0 & H_1 & \cdots & H_{n-1} \\ \vdots & \vdots & & \vdots \\ H_{1-l} & H_{2-l} & \cdots & H_{n-l} \end{bmatrix} \tag{8}$$

The number of time steps l for which outputs are specified after t_f determines the accuracy with which the desired final state is achieved. Satisfaction of Eq. (5) is a necessary but not sufficient condition for accomplishing the control task exactly. It has been shown that the precision with which the final state of the system can be specified is limited only by the observability of the system with the given outputs.[10]

The effect of measurement noise on the accuracy of the outputs at the end of the control task has also been investigated. With the assumptions that error due to noise in the pulse response data is zero mean and uncorrelated in time and that measurement hardware is selected appropriately for pulse response signal levels, so that mean square error can be expected to be proportional to the mean square value of the signal, it is found that the expected values of the predicted outputs are equal to the correct values, and the covariance matrix for the output vector at the end of the control history is inversely proportional to the number of time steps in the control history. This reflects the averaging of error that takes place in the convolution sum. The error in pulse response data can be reduced further by repeatedly generating pulse response measurements and averaging the results, in which case the covariance matrix for a column of H is inversely proportional to the number of samples averaged.

Ordinarily the inertial properties of spacecraft are known very accurately, in comparison with modal properties. In this case, it is straightforward to determine response in rigid-body modes to pulses in the control inputs so that rigid-body mode states at t_f can be added to the vector of specified outputs $\bar{\boldsymbol{y}}_f$.

The PRBC method addresses the minimum time problem by finding the minimum number of time steps for which a control profile exists satisfying Eq. (5) and the inequality constraints corresponding to input bounds. An efficient algorithm has been developed to solve this optimization problem using linear programming methods. The algorithm initially generates approximate minimum time control histories with large time steps, so that the number of unknowns and the computational cost are reduced. These approximate results give good initial guesses for solutions with smaller time steps. The H matrix corresponding to wide pulses can be synthesized easily from narrow pulse data. For example, if square pulses are used, the $H^{q\Delta t}$ matrix for a time step size of $q\Delta t$, where q is an integer, can be obtained from the $H^{\Delta t}$ matrix for a time step size of Δt by adding together groups of q successive submatrices of $H^{\Delta t}$, as in

$$H^{q\Delta t} \equiv \left[\left(\sum_{i=1}^{q} H_i^{\Delta t}\right)\left(\sum_{i=q+1}^{2q} H_i^{\Delta t}\right)\cdots\right] \tag{9}$$

In the algorithm, q is initially taken to be a power of 2, so that minimum time control histories with time steps equal to $q\Delta t$, $q\Delta t/2$, $q\Delta t/4$, etc., are successively obtained to satisfy increasing numbers of equality constraints. The algorithm checks the accuracy of the control profile by computing outputs for several time steps after the end of the control history for which outputs have not been specified. This is done for the first time, for reference, with the first minimum time control profile found, for which only the final rigid-body mode states are specified and the largest time step size ΔT_0 equal to $q\Delta t$ is used. A vector $(\bar{\boldsymbol{e}}_{\text{unsp}})_0$ is defined as

$$(\bar{\boldsymbol{e}}_{\text{unsp}})_0 \equiv \begin{Bmatrix} \boldsymbol{e}(t_f) \\ \boldsymbol{e}(t_f + \Delta T_0) \\ \vdots \\ \boldsymbol{e}(t_f + k\Delta T_0) \end{Bmatrix} \tag{10}$$

where each vector $\boldsymbol{e}$ is the error in the output vector $\boldsymbol{y}$, k is an integer, and t_f is the final time obtained for this profile. For each new control profile obtained in the algorithm, a vector

$$\bar{\boldsymbol{e}}_{\text{unsp}} \equiv \begin{Bmatrix} \boldsymbol{e}(t_{\text{unsp}}) \\ \boldsymbol{e}(t_{\text{unsp}} + \Delta T_0) \\ \vdots \\ \boldsymbol{e}(t_{\text{unsp}} + k\Delta T_0) \end{Bmatrix} \tag{11}$$

is obtained, in which $t_{\text{unsp}} = t_{\text{last}} + \Delta T_0$, where t_{last} is the last time step for which outputs have been specified for the current profile. The ratio of norms of these two vectors

$$R \equiv \frac{\|\bar{\boldsymbol{e}}_{\text{unsp}}\|_2}{\|(\bar{\boldsymbol{e}}_{\text{unsp}})_0\|_2} \tag{12}$$

is used as an indicator of how accurately the desired final state has been achieved. Note that R can be calculated easily from pulse response data. In principle, if the control task is accomplished exactly, there will be no error in outputs at any time after the end of the control history. In implementation of the algorithm, allowable values for R must be specified for each change in time step size and for final termination of the computation.

A brief outline of the algorithm for linear systems follows.

1) Obtain the pulse response matrix H for time step Δt, from measurements of the system response. Also, given the system's inertial properties, calculate a matrix of rigid body mode pulse response H_R, for which each row corresponds to one of the specified rigid body mode states. From matrices $H^{\Delta t}$ and $H_R^{\Delta t}$, obtain $H^{2\Delta t}$, $H^{4\Delta t}$, etc., and $H_R^{2\Delta t}$, $H_R^{4\Delta t}$, etc., as described earlier. Set ΔT equal to the largest time step size for which these matrices are generated.

2) First, solve the problem for the time step size ΔT considering only rigid-body mode states.

a) Choose a number of time steps $n \geq (t_f)_{\text{rigid}}/\Delta T$, where $(t_f)_{\text{rigid}}$ is the minimum time interval in which the desired change in rigid-body mode states can be accomplished with a set of p ideal control inputs having the given input bounds. It is assumed for simplicity that n is chosen so that it is not possible to achieve the final rigid-body mode states with fewer than n time steps.

b) Check whether the linear system $H_R^{\Delta T}\boldsymbol{u} = \boldsymbol{y}_R$ has a feasible solution subject to the input bounds using linear programming methods. The upper-bounded simplex method[13] is appropriate since the unknowns are subject to both upper and lower bounds. If a feasible solution does not exist, the number of time steps must be increased. If a feasible solution exists, a control profile achieving the desired final rigid-body mode states has been found. This profile will be modified over the course of the algorithm until the final solution is obtained. The norm of the error in unspecified outputs at the final time $n\Delta T$ and at several later time steps is calculated for comparison later in the algorithm.

3) Solve minimum time problems with outputs specified for an increasing number of time steps at the end of the control profile.

a) So that outputs can be specified at the final time, set

$$\bar{H} = \begin{bmatrix} H_R^{\Delta T} \\ H^{\Delta T} \end{bmatrix}$$

Find a control profile resulting in the desired rigid-body mode states and outputs at the final time, if one exists, by obtaining a feasible solution of $\bar{H}\boldsymbol{u} = \bar{\boldsymbol{y}}_f$, where $\bar{\boldsymbol{y}}_f$ now contains both the final rigid-body mode states and the outputs at the final time. In general, outputs will be specified at the first time step of length ΔT at the end of the control history for which outputs have not yet been specified. Then the matrix $\bar{H}$ will take the form

$$\bar{H} = \begin{bmatrix} H_{R1}^{\Delta T} & \cdots & H_{Rn}^{\Delta T} \\ H_1^{\Delta T} & \cdots & H_n^{\Delta T} \\ H_0^{\Delta T} & \cdots & H_{n-1}^{\Delta T} \\ \vdots & \cdots & \vdots \end{bmatrix}$$

with a new row added each time outputs are specified for an additional time step. Similarly, a new subvector of outputs will be appended to $\bar{\boldsymbol{y}}_f$ each time outputs are specified for another time step.

b) If a feasible solution cannot be found, the number of time steps n must be increased by one. If a feasible solution exists, compute unspecified outputs at time steps at the end of the control profile, and check whether the ratio R is less than a value specified by the user for the current step size. If not, specify outputs for another time step, find a feasible control profile, and check again. If so, decrease the time step size by half.

4) To continue with a time step of half the size, set $\Delta T \leftarrow \Delta T/2$ and $n \leftarrow 2n$. The current control profile $\boldsymbol{u}$ is converted for the smaller step size by inserting, after the subvector for each time step, a duplicate copy of that subvector. Because the time step size has been halved, it becomes necessary to investigate the existence of a control profile satisfying the same set of equality constraints and the input bounds and having fewer time steps than the new value of n. The existence of a feasible control profile having $n-1$ time steps is investigated by determining whether the control inputs for the first time step can be driven to zero in a feasible solution. If so, the question of whether the number of time steps can be reduced even further is addressed in the same manner. If not, a minimal number of time steps has been found for which a feasible control profile with the time step size ΔT exists. At this point, the number of time steps for which outputs are specified is again increased, starting with the time $t_f + \Delta T$, since outputs have already been specified for t_f, $t_f + 2\Delta T$, $t_f + 4\Delta T$, etc. This continues until the ratio R decreases below the value specified for the current step size. Then the time step size is halved again.

5) When the time step size Δt is reached, the minimum time control profile can be approximated with the maximum accuracy permitted by the pulse response data. The number of time steps for which outputs are specified is increased until the ratio R decreases below the value specified for the final result. The solution procedure is terminated at this point, since the desired final state of the system has been obtained with the specified accuracy.

The algorithm is demonstrated in the Numerical Example section of this paper.

PRBC for Nonlinear Maneuver Problems

The PRBC approach described is appropriate for distributed parameter systems whose behavior is linear. However, for flexible structures undergoing rapid large-angle rotations, nonlinear behavior associated with rotational motion will be significant and must be taken into account. The pulse response measurements essentially represent a linearization about the state of the system when the pulses are applied, and so nonlinear behavior will result in a deviation from this linearization. The basic approach for dealing with the nonlinearity is to use the pulse response measurements to obtain an initial approximation of the minimum time control profile and the minimum time trajectory for the system, and then to correct the pulse response measurements so that they represent a linearization about this trajectory, at least in those modes that exhibit significant nonlinear behavior. Then the minimum time problem can be solved again with corrected pulse response data. If the new minimum time trajectory differs significantly from the one based on the original pulse response data, the data can be corrected again so that a more accurate control profile can be found. In this section, the procedure for correcting pulse response measurements so that they represent a linearization about a given trajectory is presented.

For a deformable body that can undergo large rigid body displacements and small elastic deformations, the instantaneous position vector of any point on the body is given by

$$\boldsymbol{R} = \boldsymbol{R}_0 + \boldsymbol{r} + \boldsymbol{d} \tag{13}$$

where $\boldsymbol{R}_0$ locates the origin of a reference frame and is expressed in terms of an inertial frame, $\boldsymbol{r}$ is the position vector from this origin to the point in the undeformed position, and $\boldsymbol{d}$ is the elastic displacement. If the vectors $\boldsymbol{r}$ and $\boldsymbol{d}$ are expressed in terms of a frame that rotates with angular velocity $\boldsymbol{\omega}$, the kinetic energy of the system is given by

$$\begin{aligned} T = {} & \frac{1}{2} M \dot{\boldsymbol{R}}_0 \cdot \dot{\boldsymbol{R}}_0 + \frac{1}{2}\int_M \dot{\boldsymbol{d}} \cdot \dot{\boldsymbol{d}}\, \mathrm{d}m + \frac{1}{2}\int_M [\boldsymbol{\omega} \times (\boldsymbol{r} + \boldsymbol{d})] \\ & \cdot [\boldsymbol{\omega} \times (\boldsymbol{r} + \boldsymbol{d})]\, \mathrm{d}m + \dot{\boldsymbol{R}}_0 \cdot \boldsymbol{\omega} \times \int_M (\boldsymbol{r} + \boldsymbol{d})\, \mathrm{d}m \\ & + \dot{\boldsymbol{R}}_0 \cdot \int_M \dot{\boldsymbol{d}}\, \mathrm{d}m + \boldsymbol{\omega} \cdot \int_M (\boldsymbol{r} + \boldsymbol{d}) \times \dot{\boldsymbol{d}}\, \mathrm{d}m \end{aligned} \tag{14}$$

where M is the total mass of the structure. It is advantageous to use a floating reference frame for the flexible structure to simplify the equations of motion. Using the Tisserand frame,[14] which is subject to the constraints

$$\int_M (\boldsymbol{r} + \boldsymbol{d})\, \mathrm{d}m = 0 \qquad \int_M (\boldsymbol{r} + \boldsymbol{d}) \times \dot{\boldsymbol{d}}\, \mathrm{d}m = 0 \tag{15}$$

the first of which merely requires the reference frame to be centroidal, the expression for the kinetic energy becomes, after some manipulation,

$$\begin{aligned} T = {} & \frac{1}{2} M \dot{\boldsymbol{R}}_c \cdot \dot{\boldsymbol{R}}_c + \frac{1}{2}\int_M \dot{\boldsymbol{d}} \cdot \dot{\boldsymbol{d}}\, \mathrm{d}m + \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\omega} \int_M (\boldsymbol{r} + \boldsymbol{d}) \\ & \cdot (\boldsymbol{r} + \boldsymbol{d})\, \mathrm{d}m - \frac{1}{2}\int_M [\boldsymbol{\omega} \cdot (\boldsymbol{r} + \boldsymbol{d})]^2\, \mathrm{d}m \end{aligned} \tag{16}$$

The origin of the reference frame is now located at the instantaneous center of mass of the structure, which has $\boldsymbol{R}_c$ as its position vector.

For clarity, PRBC is applied to a simple example problem in which a uniform Bernoulli–Euler beam is to be rotated through a large angle in minimum time. For this beam, the transverse elastic displacement is represented as $v(x, t)$, and the vectors $\boldsymbol{r}$, $\boldsymbol{\omega}$, and $\boldsymbol{d}$ are written in terms of unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ as $\boldsymbol{r} = x\boldsymbol{i}$, $\boldsymbol{\omega} = \dot{\theta}\boldsymbol{k}$, and

$$\boldsymbol{d}(x, t) = v(x, t)\boldsymbol{j} - \frac{1}{2}\int_0^x v'(\xi, t)^2\, \mathrm{d}\xi \boldsymbol{i} \tag{17}$$

where only up to quadratic terms in $v(x, t)$ and its derivatives are retained. The kinetic and potential energies for the system become

$$\begin{aligned} T = {} & \frac{1}{2} m L \dot{\boldsymbol{R}}_c \cdot \dot{\boldsymbol{R}}_c + \frac{1}{2}\int_{-L/2}^{L/2} m \bigg[(x\dot{\theta})^2 + (v\dot{\theta})^2 + \dot{v}^2 \\ & - \left(\frac{(L/2)^2 - x^2}{2} \right) \dot{\theta}^2 v'^2 \bigg]\, \mathrm{d}x \end{aligned} \tag{18}$$

and

$$V = \frac{1}{2}\int_{-L/2}^{L/2} E I v''^2\, \mathrm{d}x \tag{19}$$

where m is the mass per unit length, EI the flexural rigidity, and L the length of the beam. The beam is controlled by transverse force inputs at its ends, so that the nonconservative virtual work, linearized in $v(x, t)$, is

$$\begin{aligned} \delta W_{\mathrm{nc}} = {} & (F_1 + F_2)\boldsymbol{j} \cdot \delta \boldsymbol{R}_c + F_1 \delta v(L/2, t) \\ & + F_2 \delta v(-L/2, t) + (F_1 - F_2)(L/2)\delta\theta \end{aligned} \tag{20}$$

Applying the extended Hamilton's principle results in ordinary differential equations for translation and rotation and a partial differential equation governing the flexible response. For a pure rotational maneuver the translational equations of motion are not needed, and the control inputs must be related by

$$F_1(t) = -F_2(t) \equiv u(t) \tag{21}$$

Because of the choice of the Tisserand frame the equation governing the rotation angle θ is uncoupled from the flexible motion and is given by

$$(mL^3/12)\ddot{\theta} = Lu \tag{22}$$

Once the minimum time control problem associated with a given H matrix has been solved for a control profile $u(t)$, this equation can be integrated to obtain $\dot{\theta}(t)$, which appears in the equation governing the flexible response and must be used to correct the pulse response measurements. The flexible response is governed by the partial differential equation

$$m\ddot{v} - (m\dot{\theta}^2/2)\{[(L^2/4) - x^2]v'\}' + EIv'''' - m\dot{\theta}^2 v = 0 \tag{23}$$

with boundary conditions

$$EIv''|_{x=\pm L/2} = 0, \qquad EIv'''|_{x=\pm L/2} = -u(t) \tag{24}$$

The terms involving $\dot{\theta}^2$ in Eq. (23) represent the nonlinear effects. The control inputs are subject to the bounds $|u(t)| \le B$, and the control task is to carry out a rest-to-rest rotational maneuver in minimum time, with no residual energy at the end of the task. Hence, the initial and final conditions become

$$\theta(0) = \dot{\theta}(0) = \dot{\theta}(t_f) = 0, \qquad \theta(t_f) = \theta_0 \tag{25}$$

and

$$v(x, 0) = \dot{v}(x, 0) = v(x, t_f) = \dot{v}(x, t_f) = 0 \qquad -L/2 < x < L/2 \tag{26}$$

The modes of a free-free uniform beam can be used to discretize the partial differential equation governing the flexible motion. The symmetric modes need not be considered due to the anti-symmetric nature of the problem. The mass-normalized antisymmetric flexible modes are given by

$$\phi_r(x) = \sqrt{\frac{2}{mL}} \left[\frac{\sinh(\beta_r L/2)\sin\beta_r x + \sin(\beta_r L/2)\sinh\beta_r x}{[\sinh^2(\beta_r L/2) - \sin^2(\beta_r L/2)]^{1/2}} \right] \qquad r = 1, 2, \ldots \tag{27}$$

where the β_r are roots of the characteristic equation $\cos\beta_r L \cosh\beta_r L = 1$ associated with the antisymmetric modes. The displacement $v(x, t)$ can now be written as

$$v(x, t) = \sum_{r=1}^{\infty} \phi_r(x)\eta_r(t) \tag{28}$$

Note that the choice of the Tisserand reference frame makes the flexible displacement $v(x, t)$ orthogonal to rigid-body motion due to Eqs. (15).

For nonlinear systems the matrix H contains measurements of the system response to pulse inputs and represents the response linearized about the state of the system when the pulses are applied. The modification of H so that it represents response linearized about a maneuver trajectory can be approximated by considering the nonlinear response of a finite number of modes. Modal equations of motion are obtained by inserting the representation of Eq. (28) in the partial differential equation of motion and exploiting the orthogonality properties of the modes. The modal equations of motion are given by

$$\ddot{\eta}_r + \omega_r^2\eta_r - \dot{\theta}^2\eta_r + \dot{\theta}^2\sum_{s=1}^{p} k_{rs}\eta_s = 2u\phi_r\left(\frac{L}{2}\right) \qquad r = 1, \ldots, p \tag{29}$$

where p is the number of modes used to model the nonlinear effects and the geometric stiffening constants k_{rs} are given by

$$k_{rs} = \frac{1}{2}\int_{-L/2}^{L/2} m\left(\frac{L^2}{4} - x^2\right)\phi_r'\phi_s'\,\mathrm{d}x \tag{30}$$

Using the flexible mode state vector $\boldsymbol{x} = [\eta_1 \ \cdots \ \eta_p \dot{\eta}_1 \ \cdots \ \dot{\eta}_p]^T$, state equations can be written as

$$\dot{\boldsymbol{x}} = \left(A_1 + \dot{\theta}^2 A_2\right)\boldsymbol{x} + Bu \tag{31}$$

in which

$$A_1 \equiv \begin{bmatrix} 0 & I \\ -\mathrm{diag}(\omega_r^2) & 0 \end{bmatrix}, \qquad A_2 \equiv \begin{bmatrix} 0 & 0 \\ I - K & 0 \end{bmatrix} \tag{32}$$

and

$$B \equiv \begin{bmatrix} \boldsymbol{0}^T & 2\phi_1(L/2) & \cdots & 2\phi_p(L/2) \end{bmatrix}^T \tag{33}$$

If the control input is piecewise constant, the solution of Eq. (31) is given by

$$\begin{aligned} \boldsymbol{x}(i\Delta t) &= \Phi[i\Delta t, (i-1)\Delta t]\,\boldsymbol{x}[(i-1)\Delta t] + \Gamma[i\Delta t, (i-1)\Delta t]\,u_i \\ &= \Phi_i\boldsymbol{x}_{i-1} + \Gamma_i u_i \end{aligned} \tag{34}$$

where the state transition matrix Φ and the input matrix Γ have the usual definitions. From the state equations, it is evident that Φ and Γ depend on the angular velocity $\dot{\theta}$, which varies slowly so that these matrices can be approximated by taking $\dot{\theta}$ to be equal to $\dot{\theta}[(i-1)\Delta t]$ over the ith time step. It is convenient to parametrize them in terms of $\dot{\theta}$ so that

$$\Phi_i \approx \Phi\{\dot{\theta}[(i-1)\Delta t]\}, \qquad \Gamma_i \approx \Gamma\{\dot{\theta}[(i-1)\Delta t]\} \tag{35}$$

Denoting the contribution to the output vector $\boldsymbol{y}$ from the first p antisymmetric flexible modes by $\boldsymbol{y}_p = C\boldsymbol{x}$, the linear relationship between u_n and $\boldsymbol{y}_p(n\Delta t)$ is obtained using

$$\boldsymbol{x}_n = \Phi_n\boldsymbol{x}_{n-1} + \Gamma_n u_n \tag{36}$$

so that

$$\boldsymbol{y}_p(n\Delta t) = C(\Phi_n\boldsymbol{x}_{n-1} + \Gamma_n u_n) \tag{37}$$

With this in mind, a vector $\boldsymbol{h}_n^{\mathrm{NL}}$ taking nonlinear behavior into account and representing a linearization about the trajectory in $\dot{\theta}$ is available from

$$\boldsymbol{h}_n^{\mathrm{NL}} = \boldsymbol{h}_n^L + C\left(\Gamma_n - \Gamma^L\right) \tag{38}$$

Here, $\boldsymbol{h}_n^L$ is from the original pulse response measurements, and so it represents a linearization about the state of the system when pulse response measurements were obtained, and Γ^L is the input matrix evaluated using the angular velocity of the system when pulses in control inputs were applied. Working backward from the final time $n\Delta t$, the state vector at $(n-1)\Delta t$ can be written as

$$\boldsymbol{x}_{n-1} = \Phi_{n-1}\boldsymbol{x}_{n-2} + \Gamma_{n-1}u_{n-1} \tag{39}$$

and, hence, the vector $\boldsymbol{h}_{n-1}^{\mathrm{NL}}$ is given by

$$\boldsymbol{h}_{n-1}^{\mathrm{NL}} = \boldsymbol{h}_{n-1}^L + C\left(\Phi_n\Gamma_{n-1} - \Phi^L\Gamma^L\right) \tag{40}$$

where Φ^L is the state transition matrix evaluated with the angular velocity equal to that of the system when pulses in control inputs were applied. In general, the vector $\boldsymbol{h}_{n-i}^{\mathrm{NL}}$ is given by

$$\boldsymbol{h}_{n-i}^{\mathrm{NL}} = \boldsymbol{h}_{n-i}^L + C\left[\left(\prod_{j=0}^{i-1}\Phi_{n-j}\right)\Gamma_{n-i} - (\Phi^L)^i\Gamma^L\right] \tag{41}$$

Generalization of these results for the multi-input case in which the vector $\boldsymbol{h}_{n-i}^{\mathrm{NL}}$ is replaced by the submatrix H_{n-i}^{NL} is straightforward.

The algorithm described in the preceding section can be used to solve the minimum time problem of a nonlinear system by correcting the H matrix as the algorithm proceeds. The H matrix can be corrected whenever the algorithm switches from a larger time step to a smaller time step, using the last control profile obtained with the larger time step. The angular velocity history used to make this correction can be expected to approach the true angular velocity profile as the size of the time step is reduced. The steps followed in making the nonlinear correction are illustrated with a numerical example.

Numerical Example

The preceding section describes the minimum time maneuver problem for large angle rotation of a flexible slender beam by means of transverse force inputs at the ends of the beam. In this section, the solution of this problem is obtained using PRBC. The exact solution of this minimum time control problem is not known. The angle of rotation is taken to be $\theta_0 = 90$ deg in this example. The results obtained are valid for uniform beams of various lengths and cross-sectional properties as long as their deformation does not violate the Bernoulli–Euler assumptions.

The bounds for the control inputs are of the form $|u(t)| \le B$. The ratio of the input bound to the angle of rotation B/θ_0 is chosen based on the time required to carry out the desired maneuver on a rigid uniform bar with the same mass. For a rigid bar, the minimum time control is bang-bang, and the final time $(t_f)_{\mathrm{rigid}}$ can be related to the ratio B/θ_0 by considering the angle at midmaneuver, which must be

$$\frac{\theta_0}{2} = \frac{1}{2}\frac{BL}{mL^3/12}\left(\frac{(t_f)_{\mathrm{rigid}}}{2}\right)^2 \tag{42}$$

If the bang-bang minimum time solution for the rigid system is applied to the flexible system, the residual energy at the end of the control history will depend on how small $(t_f)_{\mathrm{rigid}}$ is compared to the natural periods of the system. A time scale characterizing the dynamics of the system is taken to be the period T of the lowest (symmetric) flexible mode. The ratio B/θ_0 is chosen so that $(t_f)_{\mathrm{rigid}}$ is equal to $T/2$. Hence, an extremely rapid maneuver is desired, and if the bang-bang control profile for the rigid system were applied to the flexible beam, a very large amount of residual vibration could be expected. Note that the minimum time for control of a uniform second-order one-dimensional system with unbounded inputs at its two ends is equal to one-half of the period of the first flexible mode.[9]

The state transition matrix of the system whose state equations appear in Eq. (31) can be obtained using Kinariwala's procedure.[15] If $\dot{\theta}^2 A_2$ can be regarded as a perturbation of A_1, then the state transition matrix can be approximated as

$$\Phi(t + \Delta t, t) \approx \Phi_1(t + \Delta t, t) + \dot{\theta}^2(t)\Phi_2(t + \Delta t, t) \tag{43}$$

where $\Phi_1 = e^{A_1\Delta t}$, and Φ_2 can be obtained using

$$\Phi_2(t + \Delta t, t) = \int_t^{t+\Delta t} \Phi_1(t + \Delta t, \tau)A_2\Phi_1(\tau, t)\,\mathrm{d}\tau \tag{44}$$

If the control input is piecewise constant, the solution of Eq. (31) can be approximated by

$$\boldsymbol{x}(t + \Delta t) \approx \left(\Phi_1 + \dot{\theta}^2\Phi_2\right)\boldsymbol{x}(t) + \left(\Gamma_1 + \dot{\theta}^2\Gamma_2\right)\boldsymbol{u} \tag{45}$$

where the matrices Γ_1 and Γ_2 are available from

$$\Gamma_1 = \int_t^{t+\Delta t} \Phi_1(\tau, t) B \, d\tau, \qquad \Gamma_2 = \int_t^{t+\Delta t} \Phi_2(\tau, t) B \, d\tau \tag{46}$$

The accuracy with which the final state of the system approximates the desired final state is measured in terms of the residual energy at the end of the control task, given by the modal sum

$$E_{\text{res}} = \frac{1}{2}\frac{mL^3}{12}\dot{\theta}^2(t_f) + \frac{1}{2}\sum_{r=1}^{\infty}\left[\dot{\eta}_r^2(t_f) + \lambda_r \eta_r^2(t_f)\right] \tag{47}$$

The residual energy is normalized against the maximum energy in the rigid bar's minimum time control, which occurs at midmaneuver, and is given by

$$E_{m-m} = \frac{1}{2}\frac{mL^3}{12}\left(\frac{2\theta_0}{(t_f)_{\text{rigid}}}\right)^2 \tag{48}$$

The final rigid-body mode states are specified along with outputs specified in $\bar{y}_f$, so that there will be no error in these states. As a result, the first term in the residual energy will be zero. Also, for this example, displacement and velocity sensors are taken to be located at one end of the beam. Only one end of the beam needs to be considered because only antisymmetric modes are excited by the antisymmetric control. Hence, $\bar{y}_f$ will contain displacement and velocity at one end of the beam at time steps at the final time t_f and afterward.

Although it might be obtained experimentally in practice, for this example the pulse response matrix H of the system is generated with $\dot{\theta}(t) = 0$ using a modal simulation. As many as 100 antisymmetric modes are included in Eq. (28) to ensure that the effects of modal truncation will be negligible. An approximate solution of the linear problem associated with this pulse response matrix is obtained using a time step size of $\Delta T = 8\Delta t$, where $\Delta t = (t_f)_{\text{rigid}}/160$ is the smallest time step that will be used in the solution process. The first approximate solution is obtained using the algorithm for linear problems that was described earlier. Initially, only the final states for the rigid-body mode are specified, so that the minimum time control profile is bang-bang. Then, displacement and velocity outputs for one end of the beam are specified for the final time t_f and for additional time steps after the end of the control profile. Time steps for which outputs are specified are added until the ratio of errors in unspecified outputs R from Eq. (12) decreases to less than 20%, a level selected arbitrarily for this stage in the solution process.

The control profile obtained is then used to compute the angular velocity history using Eq. (22). This angular velocity history is shown in Fig. 1 as trajectory A. Based on this trajectory, Φ_i and Γ_i matrices are computed using Eq. (35), and these are used to correct individual columns of $H^{4\Delta t}$ using Eq. (41) so that the solution can proceed with the smaller time step $\Delta T = 4\Delta t$. For this example,

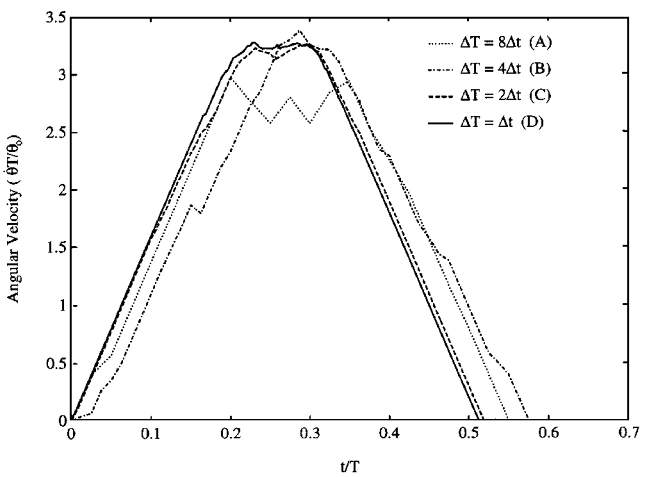

Fig. 1 Angular velocity profiles computed as ΔT is reduced.

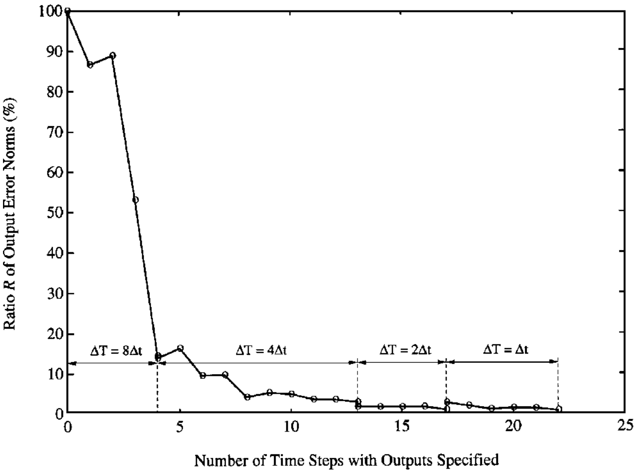

Fig. 2 Reduction in ratio of error norms as algorithm proceeds.

nonlinear effects in the first two antisymmetric flexible modes are computed to correct the pulse response data.

The minimum time problem associated with the corrected H matrix is now solved approximately using a time step size of $\Delta T = 4\Delta t$. Additional outputs are specified until R is driven to less than 4% of its original bang-bang value, i.e., 20% of the value allowed at the termination of computation with $\Delta T = 8\Delta t$. The resulting angular velocity profile is shown in Fig. 1 as trajectory B. The difference between trajectories A and B is nonnegligible. The $H^{2\Delta t}$ matrix is then corrected so that the pulse response data represents linearization about trajectory B, and the minimum time maneuver problem is solved with a time step size $2\Delta t$. Finally, the matrix $H^{\Delta t}$ is corrected and the minimum time problem is solved for the last time.

Trajectories C and D obtained as a result of the last two corrections are nearly identical, indicating that trajectory D can be taken to be a good approximation of the minimum time trajectory. The minimum time for the control task is found to be $t_f = 0.5125T$, once outputs have been specified for 21 time steps after t_f. From Fig. 1, the maximum angular velocity is about $3.3\theta_0/T$. Because $\theta_0 = \pi/2$ in this example, the maximum velocity of the rigid-body mode is about $0.8\omega_1$, where ω_1 is the natural frequency for the first (symmetric) bending mode of the beam.

It is illuminating to compare the manner in which R, the ratio of norms of error in unspecified outputs, and the residual energy in the nonlinear system at the end of the maneuver decrease over the course of the solution process. Figure 2 shows how R varies over the course of the algorithm. Two points are plotted for the same number of specified outputs every time the time step size is halved. The first value of R is obtained with an H matrix that has not yet been corrected for linearization about a new trajectory, and the second value is obtained with a corrected H matrix.

Figure 3 shows how the residual energy decreases over the course of the computation. Before the H matrix is corrected the first time, specifying additional outputs is not effective in reducing residual energy. After the first correction, though, the correspondence between R and the residual energy at the end of the maneuver is much better. When nonlinear behavior in two flexible modes is taken into account, the residual energy for the nonlinear system is ultimately driven to less than 0.2% of the midmaneuver energy for minimum time control of a rigid bar.

For comparison, if nonlinear behavior in only one mode is taken into account in correcting H, the residual energy is driven to 0.8% of midmaneuver energy. If the original pulse response data are used to obtain a minimum time control profile with no correction for nonlinear effects, the residual energy in the nonlinear system, after halving the time step size several times and specifying outputs for a number of time steps as in the example presented, is equal to 20.2% of midmaneuver energy. This is a substantial increase over the residual energy of 13.9% that results from applying the rigid bar's bang-bang minimum time control profile to the flexible system. This demonstrates that the maneuver considered in this example

Table 1 CPU time for each time step size

Time step size ΔT	Number of steps in control history	CPU time,[a] s
$8\Delta t$	22	0.14
$4\Delta t$	46	8.47
$2\Delta t$	83	16.13
Δt	164	40.56
Total CPU time		65.30

[a]CPU time for 125-MHz HyperSPARC processor.

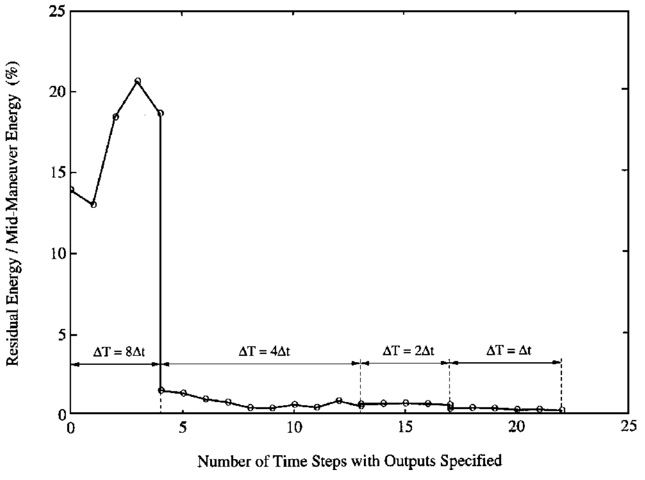

Fig. 3 Variation in residual energy over course of algorithm.

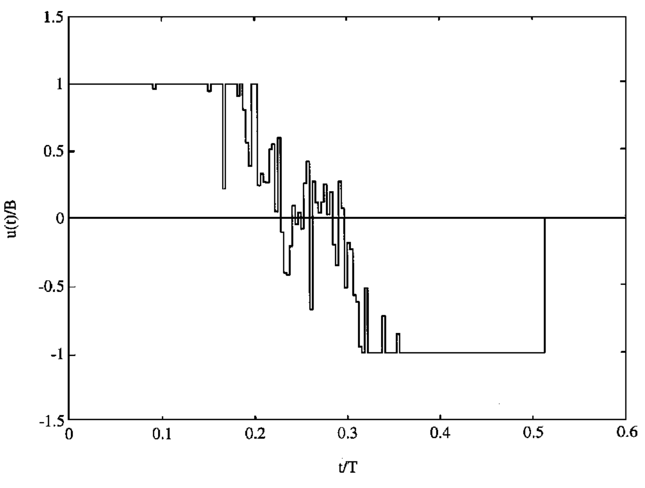

Fig. 4 Minimum time control profile for nonlinear maneuver.

problem is rapid enough that correcting for nonlinear behavior is essential.

The minimum time control profile is shown in Fig. 4. The control inputs are equal to an upper or lower bound for most but not all of the control history. This is typical of minimum time profiles for flexible systems, in contrast to finite-order systems for which control inputs are at the bounds over the entire control history.

Figure 5 shows the displacement of the beam during the rotational maneuver at intervals of $0.2t_f$. The dotted lines indicate the position of the Tisserand frame. The first antisymmetric flexible mode dominates the flexible motion of the beam during the maneuver. However, at the end of the maneuver, the third mode dominates the flexible behavior of the beam, contributing as much as 74% of the final residual energy. This is because nonlinear effects in only the first two modes have been considered in correcting the pulse response measurements.

Table 1 shows how much computation time is spent approximating the minimum time control profile with each time step size. The number of time steps in the control history is for the final result obtained with each time step size. The CPU times indicate that the total amount of computation required is quite modest.

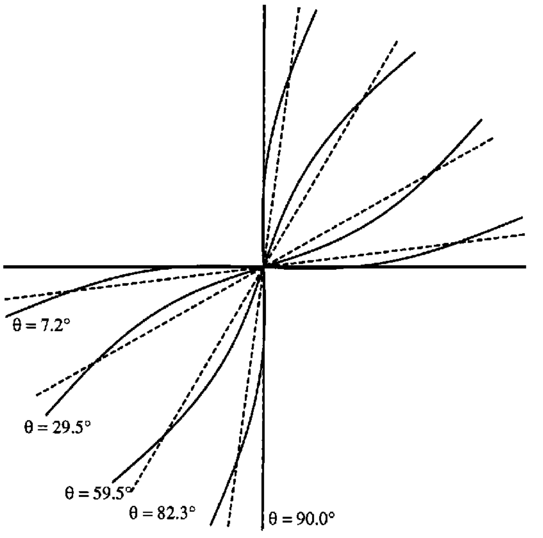

Fig. 5 Deformed beam and reference frame orientation during maneuver.

Since the amount of residual energy at the end of the maneuver is very low, and the algorithm used to solve the minimum time problems ensures that a control profile satisfying the specified constraints with fewer time steps does not exist, a very accurate solution of the minimum time maneuver problem has been found.

Since the maneuver is accomplished here using open-loop control, a natural question is whether feedback control can be implemented easily with PRBC, to deal with noise and unmodeled nonlinear behavior and to preferably introduce some desirable stability properties. Another question is whether implementation of PRBC on a more complex structure will introduce significant complications. A third question is whether PRBC can be implemented with much less computation required before a maneuver can be initiated. Reference 16 presents results of applying PRBC to a more complex system, with stable feedback control and negligible computation required before a maneuver can begin.

Conclusions

A method is presented for applying PRBC to minimum time flexible maneuver involving nonlinear behavior associated with rapid large-angle rotation. Advantages of this approach are that linear behavior is represented in terms of pulse response measurements without modal truncation or a need for a state-variable representation, and it is not necessary to solve a two-point boundary value problem. For maneuver problems, the nonlinearity is handled by correcting pulse response measurements to account for nonlinear behavior in a few flexible modes. The control profile obtained using PRBC is not bang-bang, which is appropriate since the exact minimum time control profile for the distributed system is not ordinarily bang-bang.

The numerical example demonstrates that the number of modes for which nonlinear behavior must be considered can be very small, even for a very accurate solution for an extremely rapid maneuver. This means that identification requirements can be very low when this method is used. As mentioned at the end of the last section, more recent work has addressed implementation of PRBC on more complex systems, combined with stable feedback control.

Acknowledgment

This research is sponsored by the U.S. Air Force Office of Scientific Research, U.S. Air Force Systems Command, U.S. Air Force, under Grant AFOSR-90-0297.

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